Question 1. Use ε-$\mathbb{N}$ notation to show that $\lim_{n\to\infty} \frac{2n^2}{n^2-n+1} = 2$.

Solution. Note that for $n > 1$,

$$\left| \frac{2n^2}{n^2-n+1} - 2 \right| = \frac{2n-2}{n^2-n+1} \leq \frac{2n-2}{n^2-n} = \frac{2(n-1)}{n(n-1)} = \frac{2}{n}.$$

Let $\varepsilon > 0$. By Archimedean Property, there exists $N \in \mathbb{N}$ such that

$$N > \max\left\{1, \frac{2}{\varepsilon}\right\}.$$

In particular, we have $N > 1$ and $N > 2/\varepsilon$. Hence whenever $n \geq N$,

$$\left| \frac{2n^2}{n^2-n+1} - 2 \right| \leq \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows.

Question 2. Let (x_n) be a convergent sequence of non-zero numbers. Show that if $\lim x_n \neq 0$, then the sequence $(1/x_n)$ is also convergent.
Give an example to show that the above assertion is not true if the assumption "$\lim x_n \neq 0$" is removed.

Solution. Let $L = \lim x_n \neq 0$. We claim that the sequence $(1/x_n)$ converges to $1/L$. For, first note that

$$\left| \frac{1}{x_n} - \frac{1}{L} \right| = \left| \frac{L - x_n}{x_n \cdot L} \right| = \frac{1}{|x_n||L|}|x_n - L|, \quad \forall n \in \mathbb{N}.$$

Since $L = \lim x_n$, there exists $N_1 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{|L|}{2}, \quad \forall n \geq N_1.$$

By the reverse triangle inequality, we have

$$|L| - |x_n| \leq ||x_n| - |L|| \leq |x_n - L|, \quad \forall n \in \mathbb{N}.$$

Hence

$$|L| - |x_n| < \frac{|L|}{2}, \quad \forall n \geq N_1.$$

It follows that

$$|x_n| > \frac{|L|}{2}, \quad \forall n \geq N_1.$$

Let $\varepsilon > 0$. Since $L = \lim x_n$, there exists $N_2 \in \mathbb{N}$ such that

$$|x_n - L| < \frac{|L|^2}{2}\varepsilon, \quad \forall n \geq N_2.$$

Take $N = \max\{N_1, N_2\}$, so $N \geq N_1$ and $N \geq N_2$. Then whenever $n \geq N$,

$$\left|\frac{1}{x_n} - \frac{1}{L}\right| = \frac{1}{|x_n||L|}|x_n - L| < \frac{2}{|L|^2}|x_n - L| < \varepsilon.$$

Since $\varepsilon > 0$ it arbitrary, the claim follows and hence the sequence $(1/x_n)$ is convergent.

To show that the above assertion is not true if the assumption "$\lim x_n \neq 0$" is removed, consider the sequence $(1/n)$. Note that it is a convergent sequence of non-zero numbers with $\lim(1/n) = 0$. However, the sequence $(1/(1/n)) = (n)$ is not convergent. Therefore, this example shows that the above assertion is not true if the assumption "$\lim x_n \neq 0$" is removed.

Question 3. We say that a sequence (x_n) diverges properly to $+\infty$ if for any $M > 0$, there is a positive integer N such that $x_n > M$ for all $n \geq N$.
Let (x_n) be a sequence which is not bounded above. Show that (x_n) has a strictly increasing subsequence (x_{n_k}) which diverges properly to $+\infty$.

Solution. Let $n_1 = 1$ and consider the set

$$X_1 = \{x_n : n > n_1\}.$$

We claim that X_1 is not bounded above. For, suppose on a contrary that X_1 is bounded above, then there exists some $M_1 > 0$ such that $x_n \leq M_1$ for all $n > n_1$. If we take $M_0 = \max\{x_1, ..., x_{n_1}, M_1\}$, then

$$x_n \leq M_0, \quad \forall n \in \mathbb{N}.$$

i.e., (x_n) is bounded above, which contradicts the assumption, so the claim is asserted. Since X_1 is not bounded above, there exists some $n_2 > n_1$ such that $x_{n_2} > \max\{x_{n_1}, 1\}$. i.e.,

$$x_{n_2} > x_{n_1} \quad \text{and} \quad x_{n_2} > 1.$$

Now consider the set

$$X_2 = \{x_n : n > n_2\}.$$

In a similar manner, X_2 is not bounded above. Hence there exists some $n_3 > n_2$ such that $x_{n_3} > \max\{x_{n_1}, 2\}$. Repeating this process, we can choose a sequence of strictly increasing positive integers (n_k) such that

$$x_{n_{k+1}} > x_{n_k} \quad \text{and} \quad x_{n_{k+1}} > k, \quad \forall k \in \mathbb{N}.$$

Thus, the subsequence (x_{n_k}) is strictly increasing. It left to show that (x_{n_k}) diverges properly to $+\infty$. For, let $M > 0$. By Archimedean Property, there exists some $K \in \mathbb{N}$ such that $K > M + 1$. Then whenever $k \geq K$,

$$x_{n_k} > k - 1 \geq K - 1 > M.$$

Since $M > 0$ is arbitrary, it follows that (x_{n_k}) diverges properly to $+\infty$.

Question 4. By using the definition of Cauchy sequence, show that if a Cauchy sequence (x_n) has a convergent sequence, then the sequence (x_n) must be convergent.

Solution. Let $\varepsilon > 0$ and (x_{n_k}) be a convergent subsequence of (x_n) with limit L. Since (x_{n_k}) converges to L, there exists some $K \in \mathbb{N}$ such that

$$|x_{n_k} - L| < \frac{\varepsilon}{2}, \quad \forall k \geq K.$$

Since (x_n) is Cauchy, there exists some $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N.$$

Let $k = \max\{K, N\}$. Then $k \geq K$ and $n_k \geq k \geq N$. Then whenever $n \geq N$,

$$|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) must be convergent.